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THE ROLE OF BASE PATHS IN NIELSEN FIXED POINT THEORY

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Abstract. The Reidemeister orbit set plays a crucial role in the Nielsen type theory of periodic orbits, much as the Reidemeister set does in Nielsen fixed point theory. Extending our work on Reidemeister orbit sets, we improve our theorem for formulae for Nielsen type essential orbit numbers.

1. Introduction

Nielsen fixed point theory has been extended to a Nielsen type theory of periodic orbits [4, Section III.3]. The computation of the Nielsen number often relies on the knowledge of the Reidemeister set. Ferrario [2] made an algebraic study of the Reidemeister set in relation to an invariant normal subgroup. He obtained addition formulae for Reidemeister numbers, and applied them to the Nielsen number of fibre preserving maps. Recently we studied the Reidemeister orbit set of a group endomorphism in relation to an invariant normal subgroup, obtained addition formulae for Reidemeister orbit numbers, and as application, found addition formulae for Nielsen type essential orbit numbers of fibre preserving maps [5]. Our aim in this paper is to improve the proof of the main theorem 2.4 in [5]. We prove it without conditions on the base path.

We consider a fibre preserving map $f : E \to E$ of a Hurewicz fibration $p : E \to B$ of compact connected ANR's. It induces a map $\bar{f} : B \to B$. Let K be the kernel of the homomorphism $j_* : \pi_1(F_b) \to \pi_1(E)$ induced by the inclusion of a fiber. Denote by $EO^{(n)}(f)$ the number of essential n-orbit classes of f, and by $EO_K^{(m)}$ the number of mod K essential

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orbit classes on a fibre. Under suitable conditions, we have an addition formula of the form

$$EO^{(n)}(f) = \sum_{b\in\xi} EO_K^{(m)}(h_b),$$

where the summation runs over a set ξ of essential n-orbit representatives for $\bar{f}$, d is the depth of the essential $\bar{f}$-orbit class containing b, $m = n/d$, and $h_b : F_b \to F_b$ is a variant of $f^d|F_b$.

For the basics of Nielsen fixed point theory, the reader is referred to [1] and [4].

2. The role of base paths

Let X be a compact connected ANR(absolute neighborhood retract). Let $f : X \to X$ be a map. We denote by $\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\}$ the fixed point set of f. Two fixed points $x, y \in \mathrm{Fix}(f)$ are *Nielsen related* if there is a path λ from x to y such that $f(\lambda)$ is homotopic to λ by a homotopy keeping the end points fixed. This relation divides $\mathrm{Fix}(f)$ into a finite number of *fixed point classes* of f. The set of fixed point classes will be denoted by $\mathcal{FP}(f)$.

Let $n > 0$ be a given integer. Then f acts on the set $\mathcal{FP}(f^n)$ of n-periodic point classes of f by $\mathbf{F}_{f^n} \mapsto f(\mathbf{F}_{f^n})$. In [5], the f-orbit of a class $\mathbf{F}_{f^n}$ is called an n-orbit class, denoted by $\mathbf{F}_f^{(n)}$. The set of n-orbit classes is denoted by $\mathcal{O}^{(n)}(f)$. The length of the orbit $\mathbf{F}_f^{(n)}$ is the smallest integer $\ell > 0$ such that $\mathbf{F}_{f^n} = f^\ell(\mathbf{F}_{f^n})$. The set of essential n-orbit classes will be denoted by $\mathcal{EO}^{(n)}(f)$. The essential n-orbit number $EO^{(n)}(f)$ is the cardinality of the set $\mathcal{EO}^{(n)}(f)$.

Let x be the base point in X, and take a path w from x to $f(x)$ as the base path for f. The induced endomorphism $f_*^w : \pi_1(X, x) \to \pi_1(X, x)$ is defined by

$$f_*^w(\langle\gamma\rangle) := \langle wf(\gamma)w^{-1}\rangle \qquad \text{for any loop } \gamma \text{ at } x.$$

If w is the constant path, f_*^w will be denoted by f_*^x. For $n > 1$, we have $(f^n)_*^{w_n} = (f_*^w)^n$ if the base path for f^n is taken to be $w_n := wf(w)\cdots f^{n-1}(w)$. For the sake of convenience, denote the induced endomorphism f_*^w by φ, and $\pi_1(X, x) := \pi_X$.

Notation. Suppose G is a group. For $\alpha \in G$, let $\tau_\alpha : G \to G$ denote the conjugation defined by $\tau_\alpha(\beta) = \alpha\beta\alpha^{-1}$.

Given $\varphi : \pi_X \to \pi_X$, we have the Reidemeister left action of π_X on π_X, given by

$$\beta \cdot \alpha = \beta\alpha\varphi(\beta^{-1}).$$

The Reidemeister classes are the orbits of this action, and the set of Reidemeister classes is denoted by $\mathcal{R}(\varphi)$. The Reidemeister number of f is given by $R(f) = \sharp\mathcal{R}(\varphi)$, where $\sharp$ denotes the cardinality.

Let $n > 0$ be a given integer. Then φ acts on the Reidemeister set $\mathcal{R}(\varphi^n)$ by $[\alpha]_{\varphi^n} \overset{\varphi}{\mapsto} [\varphi(\alpha)]_{\varphi^n}$. In [5], the φ-orbit of a Reidemeister class $[\alpha]_{\varphi^n}$ is called the Reidemeister n-orbit of φ, and denoted by $[\alpha]_\varphi^{(n)}$. The Reidemeister n-orbit set of φ is the set of all such φ-orbits, denoted by $\mathcal{RO}^{(n)}(\varphi)$. The length of the orbit $[\alpha]_\varphi^{(n)}$ is the smallest integer $\ell > 0$ such that $[\alpha]_{\varphi^n} = [\varphi^\ell(\alpha)]_{\varphi^n}$.

For $m \mid n$, we have a commutative diagram of pointed sets

$$\begin{array}{ccc} \mathcal{R}(\varphi^m) & \xrightarrow{\iota_{m,n}} & \mathcal{R}(\varphi^n) \\ \downarrow & & \downarrow \\ \mathcal{RO}^{(m)}(\varphi) & \xrightarrow{\iota_{m,n}} & \mathcal{RO}^{(n)}(\varphi), \end{array}$$

where the vertical maps are projections, and the horizontal maps are induced by the level-change function $\iota_{m,n} : \pi_X \to \pi_X$ defined by

$$\iota_{m,n}(\beta) := \beta\varphi^m(\beta)\varphi^{2m}(\beta)\cdots\varphi^{n-m}(\beta).$$

Recall that an φ-orbit $[\alpha]_\varphi^{(n)} \in \mathcal{RO}^{(n)}(\varphi)$ is reducible to level h, if there exists a $[\beta]_\varphi^{(h)} \in \mathcal{RO}^{(h)}(\varphi)$ such that $\iota_{h,n}([\beta]_\varphi^{(h)}) = [\alpha]_\varphi^{(n)}$. The lowest level $d = d([\alpha]_\varphi^{(n)})$ to which $[\alpha]_\varphi^{(n)}$ reduces is its depth. A Reidemeister orbit $[\alpha]_\varphi^{(n)} \in \mathcal{RO}^{(n)}(\varphi)$ is said to have the full depth property if its depth equals its length, i.e., $d = \ell$ (see [5]).

It is well known that every fixed point class of f is assigned a Reidemeister class in $\mathcal{R}(\varphi)$, called its coordinate. We get an injection $\rho : \mathcal{FP}(f) \hookrightarrow \mathcal{R}(\varphi)$, defined by $\rho(\mathbf{A}_f) := [\langle cf(c^{-1})w^{-1}\rangle]_\varphi$ for any path c from x_0 to a point x in $\mathbf{A}_f$. Thus we also get an injection $\rho : \mathcal{O}^{(n)}(f) \hookrightarrow \mathcal{RO}^{(n)}(\varphi)$, defined by $\rho(\mathbf{A}_f^{(n)}) := [\langle cf^n(c^{-1})f^{n-1}(w^{-1})\cdots f(w^{-1})w^{-1}\rangle]_\varphi^{(n)}$ for any path c from x_0 to a point x in $\mathbf{A}_f^{(n)}$. If $\ell \mid n$ and an ℓ-orbit class $\mathbf{B}_f^{(\ell)}$ lies inside an n-orbit class $\mathbf{A}_f^{(n)}$, then their coordinates are related by

$$\rho(\mathbf{A}_f^{(n)}) = \iota_{\ell,n}(\rho(\mathbf{B}_f^{(\ell)})),$$

hence $\rho(\mathbf{A}_f^{(n)})$ is reducible to level ℓ. The *depth* of an n-orbit class $\mathbf{A}_f^{(n)}$ is defined to be the depth of its coordinate $\rho(\mathbf{A}_f^{(n)})$.

We will need the mod K version of the Nielsen theory. If K is a φ-invariant normal subgroup of π_X, then we denote the induced homomorphism on π_X/K by φ_K. We then have the set $\mathcal{RO}^{(n)}(\varphi_K)$ of Reidemeister φ_K-orbits, and the mod K essential n-orbit number $EO_K^{(n)}(f)$, that is the cardinality of the set $\mathcal{EO}_K^{(n)}(f)$ of mod K essential n-orbit classes. We also have an injection $\rho_K : \mathcal{O}_K^{(n)}(f) \hookrightarrow \mathcal{RO}^{(n)}(\varphi_K)$.

The following proposition is the main tool of this paper.

Proposition 2.1. *For the base path w_n at x as above, for any path μ from x to $f(x)$, let $\mu_n = \mu f(\mu)\cdots f^{n-1}(\mu)$. Then there is an index preserving bijection*

$$r_{\mu_n,w_n} : \mathcal{RO}^{(n)}(f_*^{\mu}) \to \mathcal{RO}^{(n)}(f_*^{w})$$

given by $r_{\mu_n,w_n}([\langle\gamma\rangle]^{(n)}_{f_*^{\mu}}) = [\langle\gamma\mu_n w_n^{-1}\rangle]^{(n)}_{f_*^{w}}$. Furthermore, we have $r_{\mu_n,w_n}\circ\rho = \rho$.

Proof. See [3] and [7]. □

The following lemma is the main tool in [5].

Lemma 2.2. ([5] Reducing Lemma 2.2.) *Suppose X is a compact connected ANR, and $f : X \to X$ is a map. Suppose $x \in \mathrm{Fix}(f^n)$ lies in an n-orbit class $\mathbf{A}_f^{(n)}$ of depth d. Then there exists a homotopy $H = \{h_t : X \to X\}_{0\le t\le 1}$ connecting $f = h_0$ and $g = h_1$, such that*

(1) $x \in \mathrm{Fix}(g^d)$.

(2) *The loop $H^n(x) = \{h_t^n(x)\}_{0\le t\le 1}$ is contractible in X.*

(3) *H equals f outside of an arbitrarily given neighborhood of the point $f^{d-1}(x)$.*

Now without conditions of base paths, we improve the theorem 2.4 in [5].

In this paper we will assume that all of our fibrations $F \hookrightarrow E \to B$ (with projection $p : E \to B$) are Hurewicz fibrations with typical fibre, E and B path-connected (see [7]). We say that $f : E \to E$ is a fibre preserving map provided there is a well-defined map $\bar{f} : B \to B$ with $pf = \bar{f}p$. When such a map exists it is unique, and when B is a path connected locally path connected space it is enough that for all $b \in B$ the restriction of f takes the fibre $F_b := p^{-1}(b)$ to another fibre. For any $b \in \mathrm{Fix}(\bar{f}^n)$, we will denote the restricted map on F_b by f_b^n. For $x \in E$ let $j : F_{p(x)} \to E$ be the inclusion and K denote the kernel of the homomorphism $j_* : \pi_1(F_{p(x)}, x) \to \pi_1(E, x)$.

In [5] we call a subset $\xi \subset \mathrm{Fix}(\bar{f}^n)$ a set of essential n-orbit representatives for $\bar{f}$ if $\xi = \{b_1, b_2, \ldots, b_k\}$ contains exactly one point from each essential n-orbit class $\mathbf{F}_{\bar{f}}^{(n)} \in \mathcal{EO}^{(n)}(\bar{f})$.

For each $b \in \xi$, let d be the depth of the essential $\bar{f}$-orbit class $\mathbf{F}_{\bar{f}}^{(n)}$ containing b. Now b and $\bar{f}^d(b)$ are in the same fixed point class of $\bar{f}^n$ (because the depth is always a multiple of the length, of the $\bar{f}$-orbit class $\mathbf{F}_{\bar{f}}^{(n)}$), but not necessarily $\bar{f}^d(b) = b$. By the Reducing Lemma, there exists a homotopy $\bar{H} = \{\bar{h}_t : B \to B\}_{t \in I}$ connecting $\bar{f} = \bar{h}_0$ to some $\bar{g} = \bar{h}_1$ such that $b \in \mathrm{Fix}(\bar{g}^d)$, and the n-orbit class of $\bar{f}$ containing b corresponds to the n-orbit class of $\bar{g}$ containing b because the trace $H^n(x)$ is contractible loop. We can do this for all $b \in \xi$ simultaneously, because the $\bar{H}$ above only changes $\bar{f}$ in a small neighborhood of the $\bar{f}$-orbit of b.

By the homotopy lifting property of the fibration p, the homotopy $\bar{H}$ in B lifts to a fibre preserving homotopy $H = \{h_t : E \to E\}_{t \in I}$ connecting $f = h_0$ to some $g = h_1$.

THEOREM 2.3. *Suppose $p : E \to B$ is a fibration of compact connected ANR's with path-connected fibres, and $f : E \to E$ is a fibre preserving map. Let $\xi = \{b_1, b_2, \ldots, b_k\}$ be a set of essential n-orbit representatives for $\bar{f}$. If $\mathrm{Fix}((\bar{f}^n)_*^{b_i}) = \{1\}$ for every $b_i \in \xi$, then we have*

$$EO^{(n)}(f) = \sum_{b_i \in \xi} EO_K^{(m_i)}(g_{b_i}^{d_i}),$$

where g is the fibre preserving map from the Reducing Lemma. K is the kernel of the homomorphism $j_* : \pi_1(F_{b_i}) \to \pi_1(E)$ induced by the inclusion of the fibre, d_i is the depth of the n-orbit class of $\bar{f}$ containing b_i, and $m_i = n/d_i$.

Note that when $b_i \in \mathrm{Fix}(\bar{f}^{d_i})$, the term $EO_K^{(m_i)}(g_{b_i}^{d_i})$ in the summation can be replaced by $EO_K^{(m_i)}(f_{b_i}^{d_i})$, because we don't need to use Reducing Lemma at b_i.

Proof. By homotopy invariance we have $EO^{(n)}(f) = EO^{(n)}(g)$. So without loss of generality (by rewriting g as f) we may assume that $b_i \in \mathrm{Fix}(\bar{f}^{d_i})$ and g is the same as f.

For each $b_i \in \xi$, let $\mathbf{F}^{(n)}_{\bar{f},i}$ be the essential n-orbit class containing it. Clearly $\mathcal{EO}^{(n)}(f) = \bigcup_i p_{\mathcal{E}}^{-1}(\mathbf{F}^{(n)}_{\bar{f},i})$. So we only need to show $|p_{\mathcal{E}}^{-1}(\mathbf{F}^{(n)}_{\bar{f},i})|$ $= EO^{(m_i)}_K(f^{d_i}_{b_i})$ when $\mathrm{Fix}((\bar{f}^n)^{b_i}_*) = \{1\}$. In the following proof we shall drop the subscript i from our notation.

For each $b = p(x) \in \xi$, since $\pi_2(B)$ is trivial, we have the short exact sequence of groups

$$1 \to \pi_1(F_b, x)/K \xrightarrow{j_*} \pi_1(E, x) \xrightarrow{p_*} \pi_1(B, b) \to 1.$$

Suppose $b \in \xi$ is in the essential n-periodic point class $\mathbf{F}_{\bar{f}^n}$ which in turn is in the essential n-orbit class $\mathbf{F}^{(n)}_{\bar{f}}$ with depth d and $m := n/d$. Since d is the depth of $\mathbf{F}^{(n)}_{\bar{f}}$, $\mathbf{F}_{\bar{f}^n}$ alone constitutes an essential m-orbit class $\mathbf{F}^{(m)}_{\bar{f}^d} \subset \mathbf{F}^{(n)}_{\bar{f}}$.

Let w be the base path for f from x to $f(x)$ in E, and $p(w) = \bar{w}$ is the base path for $\bar{f}$ from b to $\bar{f}(b)$ in B. Then $\bar{w}_n = \bar{w}\bar{f}(\bar{w}) \cdots \bar{f}^{n-1}(\bar{w})$ is the base path for $\bar{f}^n$ from b to $\bar{f}^n(b)$ in B, and so $[\langle \bar{w}_n^{-1} \rangle]^{(n)}_{\bar{f}^{\bar{w}}_*}$ is the coordinate of $\mathbf{F}^{(n)}_{\bar{f}}$. We can consider the base path $\bar{w}_n = \bar{w}_d \bar{f}^d(\bar{w}_d) \cdots (\bar{f}^d)^{m-1}(\bar{w}_d)$ for $\bar{f}^n$ as the base path for $(\bar{f}^d)^m$, then $[\langle \bar{w}_n^{-1} \rangle]^{(m)}_{\bar{f}^{\bar{w}_d}_*}$ is the coordinate of $\mathbf{F}^{(m)}_{\bar{f}^d}$.

Since the path connected fibre F_b is f^d-invariant, we can choose the base path μ for f^d_b in F_b from x to $f^d_b(x) = f^d(x)$. Then $b = p(\mu)$ is the constant path at b. By Proposition 2.1 we have an index preserving bijection

$$r_{b,\bar{w}_n} : (\mathcal{RO}^{(m)}((\bar{f}^d)^b_*), [1]^{(m)}_{(\bar{f}^d)^b_*}) \to (\mathcal{RO}^{(m)}((\bar{f}^d)^{\bar{w}_d}_*), [\langle \bar{w}_n^{-1} \rangle]^{(m)}_{(\bar{f}^d)^{\bar{w}_d}_*})$$

and $\bar{\rho} = r^{-1}_{b,\bar{w}_n} \circ \bar{\rho}$.

Let ${}_K\mathbf{F}^{(m)}_{f^d_b} := {}_K\mathbf{F} \in \mathcal{EO}^{(m)}_K(f^d_b)$ be the mod K essential orbit class containing x. Then by Proposition 2.3 in [5] we have a commutative diagram of exact sequences in the category of pointed sets:

$$\begin{array}{ccccc}
(\mathcal{EO}^{(m)}_K(f^d_b), {}_K\mathbf{F}) & \xrightarrow{j_{\mathcal{E}}} & (\mathcal{EO}^{(m)}(f^d), \mathbf{F}^{(m)}_{f^d}) & \xrightarrow{p_{\mathcal{E}}} & (\mathcal{EO}^{(m)}(\bar{f}^d), \mathbf{F}^{(m)}_{\bar{f}^d}) \\
\rho_K \downarrow & & \rho \downarrow & & \bar{\rho} \downarrow \\
(\mathcal{RO}^{(m)}((f^d_b)^{\mu}_{*/K}), \rho({}_K\mathbf{F})) & \xrightarrow{j^x_*} & (\mathcal{RO}^{(m)}((f^d)^{\mu}_*), \rho(\mathbf{F}^{(m)}_{f^d})) & \xrightarrow{p_*} & (\mathcal{RO}^{(m)}((\bar{f}^d)^b_*), [1])
\end{array}$$

where the notation $[1]$ stands for $[1]^{(m)}_{(\bar{f}^d)^b_*}$.

When $\mathrm{Fix}(((\bar{f}^d)_*^b)^m) = \mathrm{Fix}((\bar{f}^n)_*^b) = \{1\}$, [5, 1.6] tells us j_*^x is injective, and so $j_{\mathcal{E}}$ is injective. Since p_* and σ preserve essentiality, we have a commutative diagram of exact sequences in the category of pointed sets:

$$\begin{array}{ccccc} 1 \to (\mathcal{EO}_K^{(m)}(f_b^d), {}_K\mathbf{F}_{f_b^d}^{(m)}) & \overset{j_{\mathcal{E}}}{\to} & (\mathcal{EO}^{(m)}(f^d), \mathbf{F}_{f^d}^{(m)}) & \overset{p_{\mathcal{E}}}{\to} & (\mathcal{EO}^{(m)}(\bar{f}^d), \mathbf{F}_{\bar{f}^d}^{(m)}) \\ & & \sigma \downarrow & & \downarrow \bar{\sigma} \\ & & (\mathcal{EO}^{(n)}(f), \mathbf{F}_f^{(n)}) & \overset{p_{\mathcal{E}}}{\to} & (\mathcal{EO}^{(n)}(\bar{f}), \mathbf{F}_{\bar{f}}^{(n)}). \end{array}$$

When $\mathrm{Fix}(\tau_{\langle \bar{w}_n^{-1} \rangle}(\bar{f}_*^{\bar{w}})^n) = \mathrm{Fix}((\bar{f}^n)_*^b) = \{1\}$, [5, 1.12] and [5, 1.7] tell us σ restricts to a bijection from $p_{\mathcal{E}}^{-1}(\mathbf{F}_{\bar{f}^d}^{(m)})$ to $p_{\mathcal{E}}^{-1}(\mathbf{F}_{\bar{f}}^{(n)})$. We get the desired equality $|p_{\mathcal{E}}^{-1}(\mathbf{F}_{\bar{f}}^{(n)})| = EO_K^{(m)}(f_b^d)$. □

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